DERIVATION OF THE LORENTZ TRANSFORMATION AND THEIR GROUP PROPERTIES

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Abstract. In this paper, the Lorentz transformation is derived from the two postulates: the equivalence of all inertial reference frames and the invariance of the speed of light. However, the most general transformation of space and time coordinates can be derived using only the equivalence of all inertial reference frames and the symmetries of space and time. The general transformation depends on one free parameter with the dimensionality of speed, which can be then identified with the speed of light c. This derivation uses the group property of the Lorentz transformations, which means that a combination of two Lorentz transformations also belongs to the class Lorentz transformations.

This paper is organized as follows. Section (II) derivation of the Lorentz transformation. In Section (III) the study of the group property of special Lorentz transformation.

Keywords: Lorentz substitution, speed of light, space and time, inertial reference frames, special theory of relativity.

LORENTS ALMASHTIRISHLARI VA ULARNING GRUPPAVIY XUSUSIYATLARI

Annotatsiya. Ushbu maqolada Lorentz transformatsiyasi ikkita postulatdan kelib chiqadi: barcha inertial sanoq sistemalarining ekvivalentligi va yorug‘lik tezligining o‘zgarmasligi. Biroq, fazo va vaqt koordinatalarining eng umumiy o‘zgarishi faqat barcha inertial sanoq sistemalarining ekvivalentligi va fazo va vaqtning simmetriyalaridan foydalangan holda olinishi mumkin. Umumiy o‘zgarish tezlikning o‘lchamiga ega bo‘lgan bitta erkin parametrga bog‘liq bo‘lib, u keyin yorug‘lik tezligi c bilan aniqlanishi mumkin. Xulosa o‘rnida, Lorents almashtirishlarining guruh bo‘lishni isbotlash uchun ikkita Lorentz almashtirishni qo‘shilishidan foydalanadi

Ushbu maqola quyidagicha tashkil topgan: I. Kirish. (II). Lorents almashtirishining keltirib chiqarilishi bo‘limi. (III) bo‘limda maxsus Lorents almashtirishining guruhiy xususiyatini o‘rganish.

Kalit so‘zlar: Lorentz almashtirishi, yorug‘lik tezligi, fozo va vaqt, inersial sonoq sistema, maxsus nisbiylik nazariyasi.

ВЫВОД ПРЕОБРАЗОВАНИЯ ЛОРЕНЦА И ИХ ГРУППОВЫЕ СВОЙСТВА

Аннотация. В данной работе преобразование Лоренца выводится из двух постулатов: эквивалентности всех инерциальных систем отсчета и инвариантности скорости света. Однако наиболее общее преобразование пространственно-временных координат можно вывести, используя только эквивалентность всех инерциальных систем отсчета и симметрии пространства и времени. Общее преобразование зависит от одного свободного параметра с размерностью скорости, которую затем можно отождествить со скоростью света **c**. Этот вывод использует групповое свойство преобразований Лоренца, что означает, что комбинация двух преобразований Лоренца также принадлежит к классу преобразований Лоренца.

Эта статья организована следующим образом. Раздел (II) вывод преобразования Лоренца. В разделе (III) изучение группового свойства специального преобразования Лоренца.

Ключевые слова: Преобразования Лоренца, скорость света, пространство и время, инерциальная система отсчета, специальная теория относительности.

I. INTRODUCTION

All physical phenomena take place in space and time. The theory of space and time is called the Special Theory of Relativity [1]. The Lorenz transformations – LT are very useful tools to solve a lot of problems associated with the special behavior of space-time coordinates of observers moving at high-speed as predicted by Einstein's special relativity theory [2, 3]. However, many authors classify them as not transparent from a physical point of view and avoid using them directly. Some others prefer to derive these equations starting with alternative theoretical constructs or particular assumptions that occur as dogmatic as well for the beginner [2, 7].

II. Derivation of the Lorentz Transformation

The derivation can be compactly written in matrix form. However, for those not familiar with matrix notation, I also write it without matrices and also we use the following conditions: Space is uniform and isotropic, time is uniform [1-4].

Let us consider two inertial reference frames s and s'. The reference frame s' moves relative to s with velocity ϑ in along the x axis. We know that the coordinates y and z perpendicular to the velocity are the same in both reference frames: $y = y'$ and $z = z'$. So, it is sufficient to consider only transformations of the coordinates x and t from the reference frame s to $x' = f_x(x,t)$ and $t' = f_t(x,t)$ in the reference frame s'.

From translational symmetry of space and time, we conclude that the functions $f_x(x,t)$ and $f_t(x,t)$ must be linear functions. Indeed, the relative distances between

two events in one reference frame must depend only on the relative distances in another frame:

$$x_1^{`} - x_2^{`} = f_x(x_1 - x_2,\ t_1 - t_2)\ ,\ t_1^{`} - t_2^{`} = f_t(x_1 - x_2, t_1 - t_2). \qquad (1)$$

Because equation (1) must be valid for any two events, the functions $f_x(x,t)$ and $f_t(x,t)$ must be linear functions. Thus

$$x' = \eta x + \mu t\ , \qquad (2)$$

$$t' = \omega x + \psi t\ , \qquad (3)$$

where η, μ, ω and ψ are some coefficients that depend on ϑ. In matrx form, equations (2) and (3) are written as

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \eta & \mu \\ \omega & \psi \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}, \qquad (4)$$

with four unknown functions η, μ, ω and ψ of ϑ.

The origin of the reference frame s' has the coordinate $x' = 0$ and moves with velocity ϑ relative to the reference frame s, so that $x = \vartheta t$. Substituting these values into Equation (2),we find

$0 = \eta\vartheta t + \mu t \rightarrow \mu = -\vartheta\eta$. Thus, equation (2) has the form

$$x' = \eta x + \mu t = \eta(x - \vartheta t) \qquad (5)$$

so we need to find only three unknown functions η, μ, ω and ψ of ϑ.

The origin of the reference frame s has the coordinate $x = 0$ and moves with velocity $-\vartheta$ relative to the reference frame s', so that $x' = -\vartheta t'$ .Substituting these values in equations (2) and (3), we find $\psi = \eta$. Thus, Equation (3) has the form

$$t' = \omega x + \eta t = \eta(\Sigma x + t), \qquad (6)$$

where we introduced the new variable $\Sigma = \omega/\eta$.

Let us change to the more common notation $\eta = \gamma$. Then equations (5) and (6) have the form

$$x' = \gamma(x - \vartheta t)\ , \qquad (7)$$

$$t' = \gamma(\Sigma x + t) \qquad (8)$$

or in the matrix form

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\vartheta \\ \Sigma & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}, \qquad (9)$$

now we need to find only two unknown functions γ_ϑ and Σ_ϑ of ϑ.

A combination of two Lorentz transformations also must be a Lorentz transformation. Let us consider a reference frame s' moving relative to s with velocity ϑ_1 and a reference frame s'' moving relative to s' with velocity ϑ_2. Then

$$x'' = \gamma_{\vartheta_2}(x' - \vartheta_2 t'), \quad x' = \gamma_{\vartheta_1}(x - \vartheta_1 t)$$
$$t'' = \gamma_{\vartheta_2}(\Sigma_{\vartheta_2} x' + t'), \quad t' = \gamma_{\vartheta_1}(\Sigma_{\vartheta_1} x + t) \tag{10}$$

or in the matrix form

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \gamma_{\vartheta_2} \begin{bmatrix} 1 & -\vartheta_2 \\ \Sigma_{\vartheta_2} & 1 \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix},$$
$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma_{\vartheta_1} \begin{bmatrix} 1 & -\vartheta_1 \\ \Sigma_{\vartheta_1} & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}. \tag{11}$$

Substituting x' and t' from the second equation (10) into the first equation (10), we find

$$x'' = \gamma_{\vartheta_2}\{\gamma_{\vartheta_1}(x - \vartheta_1 t) - \gamma_{\vartheta_1}\vartheta_2(\Sigma_{\vartheta_1} x + t)\} = \gamma_{\vartheta_2}\gamma_{\vartheta_1}\{(1 - \Sigma_{\vartheta_1}\vartheta_2)x - (\vartheta_1 + \vartheta_2)t\}$$
$$t'' = \gamma_{\vartheta_2}\{\Sigma_{\vartheta_2}(\gamma_{\vartheta_1}(x - \vartheta_1 t) + \gamma_{\vartheta_1}(\Sigma_{\vartheta_1} x + t)\} = \gamma_{\vartheta_2}\gamma_{\vartheta_1}\{(\Sigma_{\vartheta_1} + \Sigma_{\vartheta_2})x + (1 - \Sigma_{\vartheta_2}\vartheta_1)t\} \tag{12}$$

or in the matrix form

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \gamma_{\vartheta_2}\gamma_{\vartheta_1} \begin{bmatrix} 1 & -\vartheta_2 \\ \Sigma_{\vartheta_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\vartheta_1 \\ \Sigma_{\vartheta_1} & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \gamma_{\vartheta_2}\gamma_{\vartheta_1} \begin{bmatrix} 1 - \Sigma_{\vartheta_1}\vartheta_2 & -\vartheta_1 - \vartheta_2 \\ \Sigma_{\vartheta_1} + \Sigma_{\vartheta_2} & 1 - \Sigma_{\vartheta_2}\vartheta_1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{13}$$

For a general Lorentz transformation, the coefficient in front of x in equation (7) and in front of t in equation (8) are equal, i.e. the diagonal matrix elements in equation (9) are equal. equations (12) and (13) must also satisfy this requirement:

$$1 - \Sigma_{\vartheta_1}\vartheta_2 = 1 - \Sigma_{\vartheta_2}\vartheta_1 \Rightarrow \frac{\vartheta_2}{\Sigma_{\vartheta_2}} = \frac{\vartheta_1}{\Sigma_{\vartheta_1}} \tag{14}$$

In the second equation (14), the left-land side depends only on ϑ_2 ,and the right-hand side only on ϑ_1. This equation can be satisfied $\vartheta/\Sigma_\vartheta$ is a constant a independent of velocity ϑ, i.e.

$$\Sigma_\vartheta = \frac{\vartheta}{a} \tag{15}$$

Substituting equation (15) into equations (7) and (8), as well as (9), we find

$$x' = \gamma_\vartheta(x - \vartheta t), \; t' = \gamma_\vartheta(\frac{x\vartheta}{a} + t) \tag{16}$$

or in the matrix form

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma_\vartheta \begin{bmatrix} 1 & -\vartheta \\ \vartheta/a & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{17}$$

Now we need to find only one unknown function γ_ϑ , whereas the coefficient a is a fundamental constant independent on ϑ.

Let us make the Lorentz transformation from the reference frame s to s' and then from s' back to s .The first transformation is performed with velocity ϑ, and the

second transformation with velocity $-\vartheta$. The equations are similar to equations (10) and (11):

$$x = \gamma_{-\vartheta}\left(x' + \vartheta t'\right), \quad x' = \gamma_{\vartheta}\left(x - \vartheta t\right)$$
$$t = \gamma_{-\vartheta}\left(-\frac{x'\vartheta}{a} + t'\right), \quad t' = \gamma_{\vartheta}\left(\frac{x\vartheta}{a} + t\right), \tag{18}$$

or in the matrix form

$$\begin{bmatrix} x \\ t \end{bmatrix} = \gamma_{-\vartheta}\begin{bmatrix} 1 & \vartheta \\ -\vartheta/a & 1 \end{bmatrix}\begin{bmatrix} x' \\ t' \end{bmatrix},$$
$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma_{\vartheta}\begin{bmatrix} 1 & -\vartheta \\ \vartheta/a & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix}. \tag{19}$$

Substituting x' and t' from the first equation (18) into the second one, we find

$$x = \gamma_{-\vartheta}\left\{\gamma_{\vartheta}\left(x - \vartheta t\right) + \vartheta\gamma_{\vartheta}\left(\frac{x\vartheta}{a} + t\right)\right\} = \gamma_{-\vartheta}\gamma_{\vartheta}x - \gamma_{-\vartheta}\gamma_{\vartheta}\vartheta t + \gamma_{-\vartheta}\gamma_{\vartheta}\frac{x\vartheta^2}{a} + \gamma_{-\vartheta}\gamma_{\vartheta}\vartheta t = \gamma_{-\vartheta}\gamma_{\vartheta}\left(1 + \frac{\vartheta^2}{a}\right)x,$$

$$t = \gamma_{-\vartheta}\left\{-\frac{\gamma_{\vartheta}\left(x - \vartheta t\right)\vartheta}{a} + \gamma_{\vartheta}\left(\frac{x\vartheta}{a} + t\right)\right\} = -\gamma_{-\vartheta}\gamma_{\vartheta}\frac{x\vartheta}{a} + \gamma_{-\vartheta}\gamma_{\vartheta}\frac{\vartheta^2 t}{a} + \gamma_{-\vartheta}\gamma_{\vartheta}\frac{x\vartheta}{a} + \gamma_{-\vartheta}\gamma_{\vartheta}t = \gamma_{-\vartheta}\gamma_{\vartheta}\left(1 + \frac{\vartheta^2}{a}\right). \tag{20}$$

or in the matrix form

$$\begin{bmatrix} x \\ t \end{bmatrix} = \gamma_{-\vartheta}\gamma_{\vartheta}\begin{bmatrix} 1 & \vartheta \\ -\vartheta/a & 1 \end{bmatrix}\begin{bmatrix} 1 & -\vartheta \\ \vartheta/a & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} = \gamma_{-\vartheta}\gamma_{\vartheta}\begin{bmatrix} 1+\vartheta^2/a & 0 \\ 0 & 1+\vartheta^2/a \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \tag{21}$$

equations (20) and (21) must be valid for any x and t, so

$$\gamma_{-\vartheta}\gamma_{\vartheta} = \frac{1}{1+\vartheta^2/a} \tag{22}$$

Because of the space symmetry, the function γ_{ϑ} must depend only on the absolute value of velocity ϑ, but not on its direction, so $\gamma_{-\vartheta} = \gamma_{\vartheta}$ Thus we find

$$\gamma_{\vartheta} = \frac{1}{\sqrt{1+\vartheta^2/a}} \tag{23}$$

Substituting equation (23) into equations (16) and (17), we find the final expressions for the transformation

$$x' = \frac{x - \vartheta t}{\sqrt{1+\vartheta^2/a}}, \quad t' = \frac{x\vartheta/a + t}{\sqrt{1+\vartheta^2/a}} \tag{24}$$

or in the matrix form

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \frac{1}{\sqrt{1+\vartheta^2/a}}\begin{bmatrix} 1 & -\vartheta \\ \vartheta/a & 1 \end{bmatrix}\begin{bmatrix} x \\ t \end{bmatrix} \tag{25}$$

equations (24) and (25) have one fundamental parameter a, which has the dimensionality of velocity squared.

If $a < 0$, we can write it as $a = -c^2$ (26)

Then equations (24) and (25) become the standard Lorentz transformation:

$$x' = \frac{x - \vartheta t}{\sqrt{1 - \vartheta^2 / c^2}},\ t' = \frac{-x\vartheta / c^2 + t}{\sqrt{1 - \vartheta^2 / c^2}} \tag{27}$$

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \frac{1}{\sqrt{1 - \vartheta^2 / c^2}} \begin{bmatrix} 1 & -\vartheta \\ -\vartheta / c^2 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{28}$$

It is easy to check from equation (27) that, if a particle moves with velocity c in one reference frame, it also moves with velocity c in any other reference frame, i.e. if $x = ct$ then $x' = ct$. Thus the parameter c is the invariant speed.

Knowing about the Maxwell equations and electromagnetic waves, we can identify this parameter with the speed of light. It is straightforward to check that the Lorentz transformation (27) and (28) preserves the spacetime interval

$$(ct')^2 - (x')^2 = (ct)^2 - (x)^2, \tag{29}$$

so it has the Minkowski metric[4-7].

III. Group property of special Lorentz transformation.

The Lorentz transformation (28) can be written more symmetrically as

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \frac{1}{\sqrt{1 - \vartheta^2 / c^2}} \begin{bmatrix} 1 & -\vartheta / c \\ -\vartheta / c & 1 \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix} \tag{30}$$

Instead of velocity ϑ, let us introduce a dimensionless variable α called rapidity and defined as

$$th\alpha = \vartheta / c, \quad ch^2\alpha - sh^2\alpha = 1, \quad 1 - th^2\alpha = \frac{1}{ch^2\alpha},$$

where th, sh and ch is the hyperbolic tangent, hyperbolic sine and hyperbolic cosine. We use these formulas we will transform Lorentz of a factor and thus

$$\frac{1}{\sqrt{1 - \vartheta^2 / c^2}} = \frac{1}{\sqrt{1 - th^2\alpha}} = \frac{1}{\sqrt{\frac{ch^2\alpha - sh^2\alpha}{ch^2\alpha}}} = ch\alpha$$

Then equation (30) acquires the following form:

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \begin{bmatrix} ch\alpha & -sh\alpha \\ -sh\alpha & ch\alpha \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix} \tag{31}$$

Let us consider a combination of two consecutive Lorentz transformations with velocities ϑ_1 and ϑ_2 as described in the first part. The rapidity α of the combined boost has a simple relation to the rapidities α_1 and α_2 each boost:

$$\alpha = \alpha_1 + \alpha_2 \tag{32}$$

Indeed, equation (32) represents the relativistic law of adding velocities

$$th\alpha = \frac{th\alpha_1 + th\alpha_2}{1 + th\alpha_1 th\alpha_2} \Rightarrow \vartheta = \frac{\vartheta_1 + \vartheta_2}{1 + \frac{\vartheta_1 \vartheta_2}{c^2}}. \tag{33}$$

Let us denote the 2×2 matrix in equation (31) as $M(\alpha)$. Then, the combination of two boosts has the simple matrix form

$$M(\alpha_1+\alpha_2)=M(\alpha_1)M(\alpha_2). \tag{34}$$

We see that the Lorentz transformations form a group, similar to the group of rotations, with the rapidity being the (imaginary) rotation angle.

For example, we can check the axiom of associativity.

Associative Property of Special Lorentz Transformation

Consider three inertial frames of reference s, s', s'' and s''' where the frame s is at rest and the frame s' is moving with velocity u with respect to s, s'' is moving with velocity v with respect to s', s''' is moving with velocity w respect to s'' then associative property says that $(u\oplus v)\oplus w=u\oplus(v\oplus w)$. Here we use unit $c=1$.

The velocity addition formul a for special Lorentz transformation [3] can be written as

$$w''=u\oplus v=\frac{u+v}{1+uv}. \tag{35}$$

Now, if s'' moves with velocity w'' with respect to s, then according to the velocity additi on formula for special Lorentz transformati on [3, 4] can be written as

$$w'''=(u\oplus v)\oplus w=\frac{w''+w}{1+w''w}=\frac{\dfrac{u+v}{1+uv}+w}{1+\dfrac{u+v}{1+uv}w}=\frac{u+v+w+uvw}{1+uv+uw+vw}. \tag{36}$$

Again, let s'' moves with velocity v with respect to s' and s''' moves with velocity w with respect to s'' then according to the velocity addition formula for the Special Lorentz transformati on [3-5] the resultant velocity of v and w can be written as

$$w'=v\oplus w=\frac{v+w}{1+vw}. \tag{37}$$

Finally, let s'' moves with velocity u with respect to s' and s''' moves with velocity w' with respect to s'' the resultant velocity [3] of u and w' can be written as

$$w''=u\oplus(v\oplus w)=\frac{u+w'}{1+uw'}=\frac{u+\dfrac{v+w}{1+vw}}{1+u\dfrac{v+w}{1+vw}}=\frac{u+v+w+uvw}{1+uv+uw+vw}=(u\oplus v)\oplus w. \tag{38}$$

Hence,

$$(u\oplus v)\oplus w=u\oplus(v\oplus w). \tag{39}$$

Consequently, special Lorentz transformation satisfies the associative property.

Conclusion

In conclusion, we have seen the derivation of the Lorentz transformation and the existence of the group property. As a proof, the property of associativity was checked.

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